2/EH–29 (ii) (Syllabus–2015)

2 0 1 7

(April)

MATHEMATICS

(Elective / Honours)

(Geometry and Vector Calculus)

(GHS–21)

Marks : 75

Time : 3 hours

The figures in the margin indicate full marks for the questions

Answer **five** questions, choosing **one** from each Unit

UNIT—I

1. (a) If the two pairs of lines $x^2 - 2pxy - y^2 = 0$ and $x^2 - 2qxy - y^2 = 0$ be such that each pair bisects the angles between the other pair, prove that $pq + 1 = 0$. 5

   (b) Find the angle through which a set of rectangular axes must be turned without the change of origin so that the expression $7x^2 + 4xy + 3y^2$ will be transformed into the form $a'x^2 + b'y^2$. 5

(Turn Over)

(c) Find the diameter of the conic $15x^2 - 20xy + 16y^2 = 1$ conjugate to the diameter $y + 2x = 0$. 5

2. (a) Find the lengths of the semiaxes of the conic $ax^2 + 2hxy + ay^2 = d$. 6

(b) Find the centre of the conic given by the equation

$$3x^2 - 8xy + 7y^2 - 4x + 2y - 7 = 0$$ 5

(c) Find the equation of the polar of the point (2, 3) with respect to the conic

$$x^2 + 3xy + 4y^2 - 5x + 3 = 0$$ 4

UNIT—II

3. (a) Prove that the locus of the point of intersection of the normals to the parabola $y^2 = 4ax$ at the extremities of a focal chord is the parabola $y^2 = a(x - 3a)$. 5

(b) Prove that the sum of the reciprocals of two perpendicular focal chords of a conic is constant. 5

(c) Prove that the locus of the middle point of the portion of a tangent to the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

intercepted between the axes is given by

$$\frac{a^2}{x^2} + \frac{b^2}{y^2} = 4$$ 5

4. (a) If e_1 and e_2 be the eccentricities of a hyperbola and its conjugate, show that

$$\frac{1}{e_1^2} + \frac{1}{e_2^2} = 1$$ 5

(b) Prove that two tangents can be drawn from a given point of an ellipse. 5

(c) If the tangent $y = mx + \sqrt{a^2m^2 - b^2}$ touches the hyperbola

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

at the point $(a\sec\theta, b\tan\theta)$, prove that

$$\sin\theta = \frac{b}{am}$$ 5

UNIT—III

5. (a) Find the equation of the plane passing through the point $(1, -2, 1)$ and the line of intersection of the planes $2x - y + 3z - 2 = 0$ and $x + 2y - 4z + 3 = 0$ 4

(Turn Over)

(b) Prove that the lines

$$\frac{x-1}{2}=\frac{y-2}{3}=\frac{z-3}{4} \text{ and}$$

$$4x-3y+1=0=5x-3z+2$$

are coplanar. 5

(c) Prove that the shortest distance between the lines

$$\frac{x-5}{3}=\frac{y-7}{-16}=\frac{z-3}{7} \text{ and}$$

$$\frac{x-9}{3}=\frac{y-13}{8}=\frac{z-15}{-5}$$

is 14. 6

6. (a) Find the equation of the sphere which touches the sphere $x^2+y^2+z^2=21$ at the point (1, −2, 4) and passes through the point (3, 4, 0). 5

(b) Find the equation of the cone whose vertex is (2, 2, 2) and the base is $z=0$, $x^2+y^2=36$. 5

(c) Find the equation of a right circular cylinder whose axis is

$$\frac{x-1}{2}=\frac{y-2}{-1}=\frac{z-3}{2}$$

and its radius is 5. 5

(Continued)

UNIT—IV

7. (a) If $\hat{a}, \hat{b}, \hat{c}$ be three unit vectors such that

$$\hat{a}\times(\hat{b}\times\hat{c})=\frac{1}{2}\hat{b}$$

find the angles which $\hat{a}$ makes with $\hat{b}$ and $\hat{c}$, given that $\hat{b}$ and $\hat{c}$ being non-parallel. 5

(b) Prove that

$$(\vec{b}\times\vec{c})\cdot(\vec{a}\times\vec{d})+(\vec{c}\times\vec{a})\cdot(\vec{b}\times\vec{d})+(\vec{a}\times\vec{b})\cdot(\vec{c}\times\vec{d})=0$$ 5

(c) Prove the Lagrange's identity 5

$$(\vec{a}\times\vec{b})\cdot(\vec{c}\times\vec{d})=\begin{vmatrix}\vec{a}\cdot\vec{c} & \vec{a}\cdot\vec{d}\\ \vec{b}\cdot\vec{c} & \vec{b}\cdot\vec{d}\end{vmatrix}$$

8. (a) Show that a necessary and sufficient condition for $\vec{u}(t)$ to be constant is 3

$$\frac{d\vec{u}}{dt}=0$$

(b) If $\hat{r}$ is a unit vector, show that

$$\left|\hat{r}\times\frac{d\hat{r}}{dt}\right|=\left|\frac{d\hat{r}}{dt}\right|$$ 3

(Turn Over)

(c) If $\vec{r} = \vec{a}\cos\omega t + \vec{b}\sin\omega t$, show that

(i) $\hat{r} \times \frac{d\vec{r}}{dt} = \omega\vec{a} \times \vec{b}$

(ii) $\frac{d^2\vec{r}}{dt^2} = -\omega^2\vec{r}$

where $\vec{a}$ and $\vec{b}$ are constant vectors. 2+3

(d) A particle moves along the curve $x = 4\cos t$, $y = 4\sin t$, $z = 6t$. Find the velocity and acceleration at time $t = 0$ and $t = \frac{\pi}{2}$. 4

UNIT—V

9. (a) Find the directional derivative of the function

$$f(x) = x^2 - y^2 + 2z^2$$

at the point $P(1, 2, 3)$ in the direction of the line PQ, where Q has coordinates $(5, 0, 4)$. 5

(b) Show that grad $f(r) \times \vec{r} = 0$, where

$$r = |\vec{r}| = \sqrt{x^2 + y^2 + z^2}$$ 5

(c) Determine the constants a, b, c so that the vector

$$\vec{f} = (x + 2y + az)\hat{i} + (bx - 3y - z)\hat{j} + (4x + cy + 2z)\hat{k}$$

is irrotational. 5

10. (a) Find a unit normal to the surface $\phi = 2x^2y + 3yz - 4$ at the point $(1, -1, -2)$. 5

(b) If $r = |\vec{r}|$, where $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$, prove that $\nabla \log|\vec{r}| = \frac{1}{r^2}\vec{r}$. 5

(c) Find the divergence and curl of the vector

$$\vec{f} = (x^2 - y^2)\hat{i} + 2xy\hat{j} + (y^2 - xy)\hat{k}$$ 5

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